2

NUSC Technical Report 8865
10 April 1991

AD-A235 774

Required Threshold Settings and Signal-to-Noise Ratios for Combined Normalization and Or-ing

Albert H. Nuttall
Surface ASW Directorate

A-1

Naval Underwater Systems Center
Newport, Rhode Island • New London, Connecticut

Approved for public release; distribution is unlimited.

PREFACE

This research was conducted under NUSC Project No. A70272, Subproject No. RR0000-N01, **Selected Statistical Problems in Acoustic Signal Processing**, Principal Investigator, Dr. Albert H. Nuttall (Code 304). This technical report was prepared with funds provided by the NUSC In-House Independent Research and Independent Exploratory Development Program, sponsored by the Office of the Chief of Naval Research. This report was also prepared under the Multistatic Sonar System, Project No. M20027, Program Manager CDR T. Mason, NAVSEA. The NUSC Program Manager is James W. Kyle, while the principal investigators are Ira B. Cohen and Robert T. Coia.

The technical reviewer for this report was Ira B. Cohen (Code 3332).

REVIEWED AND APPROVED: 10 APRIL 1991

for

DAVID DENCE
ATD for Surface Antisubmarine Warfare Directorate

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
10 April 1991

3. REPORT TYPE AND DATES COVERED
Progress

4. TITLE AND SUBTITLE
Required Threshold Settings and Signal-to-Noise Ratios for Combined Normalization and Or-ing

5. FUNDING NUMBERS
PE 61152N

6. AUTHOR(S)
Albert H. Nuttall

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Underwater Systems Center
New London Laboratory
New London, CT 06320

8. PERFORMING ORGANIZATION REPORT NUMBER
TR 8865

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Chief of Naval Research
Office of the Chief of Naval Research
Arlington, VA 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

The system of interest here includes the combined nonlinear transformations of normalization and or-ing in the data processing chain. The required threshold settings for specified false alarm probabilities are determined, as well as the required input signal-to-noise ratios for specified detection probabilities. Specializations to normalization or or-ing alone are available by particular choices of parameter values built into the models.

When the sizes of the transformations are changed, that is, the amount of normalization and/or or-ing, the thresholds will also have to be changed if the earlier probabilities are to be maintained; these options are included in the first program listed. In addition, the possibility of changing the number of thresholds, at the same time that the sizes of the nonlinear

14. SUBJECT TERMS
threshold setting, signal-to-noise ratio, normalization, or-ing, nonlinear, false alarm probability

15. NUMBER OF PAGES
50

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102

13. ABSTRACT (Cont'd)

transformations are changed, is included in the second program listed. Furthermore, four alternatives are allowed for selecting the spacing of the thresholds, namely, equispaced in power, equispaced in decibels, preset normalized thresholds, or preset probabilities.

14. SUBJECT TERMS (Cont'd)

detection probability moments

TABLE OF CONTENTS

LIST OF SYMBOLS

RV	random variable
IID	independent identically distributed
PDF	probability density function
CDF	cumulative distribution function
CF	characteristic function
Prob	probability
x_m	m-th noise-only random variable, (1)
M	M+1 is the number of random variables $\{x_m\}$
$p_x(u)$	probability density function of x_m, (1)
$C_x(X)$	cumulative distribution function of x_m, (1)
$f_x(\xi)$	characteristic function of x_m, (2)
$\mu_x(j)$	j-th moment of x_m
$\chi_x(j)$	j-th cumulant of x_m
$\tilde{C}_x(P)$	inverse cumulative distribution function for x, (3)
f	nonlinear transformation, (5)
y	output of nonlinear transformation, (5)
w_{km}	inputs to normalizer and or-ing device, (8)
x_k	output from or-ing device, (8)
K	size of or-ing operation, (8)
$C_y(Y)$	cumulative distribution function of y, (9)
Y_1	first threshold for application to y, (9)
P_1	specified cumulative probability at Y_1, (9)
$\tilde{C}_y(P)$	inverse cumulative distribution function for y, (9)
Y_T	last threshold for application to y, (10)
T	total number of thresholds, (10)

A_y	scale factor, (10)
dB_y	decibel ratio between Y_T and Y_1, (10)
Y_t	t-th threshold for application to y, (11)
$\underline{Y}_t$	t-th normalized threshold for y, (12)
μ_y	mean of random variable y, (12)
σ_y	standard deviation of random variable y, (12)
P_t	cumulative probability value at threshold Y_t, (13)
$1-P_t$	false alarm probability, (13)
D_t	t-th threshold in decibels, (14), (16)
N	new size of nonlinear transformation, (20)
z	output of new nonlinear transformation, (20)
$C_z(Z)$	cumulative distribution function of z, (21)
Z_t	t-th threshold for application to z, (21)
$\tilde{C}_z(P)$	inverse cumulative distribution function for z, (21)
$\underline{Z}_t$	t-th normalized threshold for z, (22)
μ_z	mean of random variable z, (22)
σ_z	standard deviation of random variable z, (22)
dB_z	decibel ratio between Z_T and Z_1, (23)
U	new number of thresholds, (24)
Z'_u	u-th new threshold, (24)
P'_u	cumulative probability at Z'_u, (25)
$\underline{Z}'_u$	u-th normalized threshold for z, (26)
$C_y(Y;R)$	cumulative distribution of y with signal present, (27)
R	input power signal-to-noise ratio, (27)
S_1	specified probability, (27)
R_t	required signal-to-noise ratio at threshold Y_t, (27)

Pd_j	j-th detection probability, (30)
a	average of M random variables, (31)
$f_a(\xi)$	characteristic function of a, (32), (36)
$p_a(u)$	probability density function of a, (32), (37)
$p_y(u)$	probability density function of y, (34), (39)
$\overline{y^j}$	j-th moment of random variable y, (41)
$(b)_j$	$b(b+1)\cdots(b+j-1)$, (41)
$\chi_y(j)$	j-th cumulant of y, (55)
a_k	k-th denominator average, (62)
F_1, F_2	auxiliary constants, (79)
b_k	k-th denominator average, (81)
L	new size of or-ing device, (82)
G_1, G_2	auxiliary constants, (86)
Q_t	auxiliary function, (95), (100)

REQUIRED THRESHOLD SETTINGS AND SIGNAL-TO-NOISE RATIOS FOR COMBINED NORMALIZATION AND OR-ING

INTRODUCTION

When weak signals of unknown strength and location have to be detected in the presence of noise of unknown and varying level, it is necessary to make estimates of the intensity of the interfering background. These estimated (noise) levels are then compared with that for a candidate signal level and location for purposes of making statements about the likelihood of signal presence or absence in that particular data segment under investigation. Here, we will investigate the performance capability of such a normalizer, in terms of the false alarm and detection probabilities, and determine the thresholds and input signal-to-noise ratios required to attain these probabilities.

Furthermore, when large amounts of multichannel data have to be processed with limited computational facilities in reasonable or short intervals of time, it is necessary to resort to shortcuts or data reduction in order to avoid overload. One possible approach is to employ or-ing, in which only the largest of a set of quantities is retained for further data processing and decision making.

Finally, in practice, it is often necessary to utilize both normalization and or-ing together. This combination of nonlinear processors requires a resetting of the thresholds that would have been appropriate for use of the normalizer alone. Here, we shall investigate all three situations, namely normalization, or-ing, and a combination of normalization and or-ing, in terms of the probabilities and thresholds stated above.

When the size of the nonlinear transformation, whether it be normalization, or-ing, or both, is changed, the required thresholds will have to be changed if the previously realized (false alarm) probabilities are to be maintained. For example, suppose we had been or-ing 8 random variables and decided to change the or-ing size to accept 16 random variables instead, for purposes of further data reduction. Then, the required threshold settings would have to be modified to maintain specified false alarm probabilities, as would the required input signal-to-noise ratios for specified detection probabilities. This maintenance of probabilities under a change of size of transformation will be investigated here.

At the same time that the size of a nonlinear transformation is changed, it may be desired to subject its output to a different number of thresholds than were utilized previously. This possibility is allowed and analyzed in addition.

The particular physical situation considered here is that of multiple simultaneous beamformer outputs, each with large banks of narrowband filters subject to envelope-squared detection. For Gaussian noise inputs, the probability density function of each of these device outputs is exponential. Furthermore, when a Gaussian signal is also present at the input of one of these narrowband filters, the corresponding probability density function of the detected output is still exponential, but with a level governed by the signal-to-noise ratio at that particular filter output. This scenario will be the mainstay of the analysis here.

The physical motivation for this study is to be able to set requantization levels on a display, in order to achieve constant marking density independent of the signal processing parameters such as the amount of or-ing and normalizer size. Such displays occur in active as well as passive sonar systems.

DEFINITIONS OF FUNCTIONS

The random variables (RV) for noise-only, $\{x_m\}$ for $0 \le m \le M$, are independent and identically distributed (IID) with common probability density function (PDF) $p_x(u)$ and cumulative distribution function (CDF) $C_x(X)$, where

$$C_x(X) = \text{Prob}(x_m < X) = \int_{-\infty}^{X} du\ p_x(u) \ . \tag{1}$$

The corresponding characteristic function (CF) of RV x_m is

$$f_x(\xi) = \int du\ \exp(i\xi u)\ p_x(u) \ , \tag{2}$$

where integrals without limits are over $-\infty,+\infty$. The moments and cumulants of order j of RV x_m are denoted by $\mu_x(j)$ and $\chi_x(j)$, respectively. The inverse function to CDF $C_x(X)$ in (1) is denoted as $\tilde{C}_x(P)$; that is,

$$\text{if } P = C_x(X), \quad \text{then } X = \tilde{C}_x(P) \quad \text{for } 0 < P < 1 \ . \tag{3}$$

As an example, consider narrowband filter outputs for which

$$p_x(u) = \exp(-u) \quad \text{for } u > 0 \ ,$$

$$C_x(X) = 1 - \exp(-X) \quad \text{for } X > 0 \ ,$$

$$f_x(\xi) = (1 - i\xi)^{-1} \quad \text{for all } \xi \ ,$$

$$\tilde{C}_x(P) = -\ln(1 - P) \quad \text{for } 0 < P < 1 \ ,$$

$$\mu_x(j) = j! \ , \quad \chi_x(j) = (j-1)! \quad \text{for } j \ge 1 \ . \tag{4}$$

The scaling of RV x_m in (4) has been taken such that its mean is 1. This is done solely for notational convenience; it will not affect the probabilities realized herein nor the required signal-to-noise ratios. However, it does influence the threshold settings calculated, which would have to be scaled for a different input noise level.

THRESHOLD RESETTING

A collection of IID RVs, $\{x_m\}$ for $0 \le m \le M$, is subject to a nonlinear transformation f, yielding output

$$y = f(x_0, x_1, \ldots, x_M) \ . \tag{5}$$

For example, a normalizer is characterized by

$$y = \frac{x_0}{\frac{1}{M}\left(x_1 + x_2 + \cdots + x_M\right)} \ , \quad M \ge 1 \ , \tag{6}$$

while an or-ing device yields

$$y = \max(x_1, x_2, \ldots, x_M) \ , \quad M \ge 1 \ . \tag{7}$$

A combination of a normalizer and or-ing device will require a more general formulation; then we would use

$$x_k = \frac{w_{ko}}{\frac{1}{M}\sum_{m=1}^{M} w_{km}} \quad \text{for } 1 \le k \le K \ ,$$

$$y = \max(x_1, x_2, \ldots, x_K) \ , \tag{8}$$

where we need two parameters, M and K, and $\{w_{km}\}$ are K(M+1) IID RVs. Of course, then the K RVs $\{x_k\}$ are also IID.

Let $C_y(Y)$ be the CDF of RV y at the output of general nonlinearity f in (5). Choose threshold Y_1 such that specified cumulative probability P_1 is realized there; that is,

$$P_1 = C_y(Y_1) \ , \quad \text{or} \quad Y_1 = \tilde{C}_y(P_1) \ , \tag{9}$$

where $\tilde{C}_y$ is the inverse function to C_y. $1-P_1$ is the false alarm probability at threshold Y_1. Also, choose additional thresholds $\{Y_t\}$ such that $Y_1 < Y_2 < \cdots < Y_T$, for a total of T thresholds, with the largest one being

$$Y_T = Y_1 \, A_y \ , \quad \text{where} \quad dB_y = 10 \log_{10}(A_y) \tag{10}$$

is a specified decibel value. Then take the remaining thresholds according to equal spacing rule

$$Y_t = Y_1 + \frac{Y_T - Y_1}{T - 1}(t - 1) \quad \text{for } 1 \le t \le T \ . \tag{11}$$

The normalized thresholds for RV y are defined as

$$\underline{Y}_t = \frac{Y_t - \mu_y}{\sigma_y} \quad \text{for } 1 \le t \le T \ . \tag{12}$$

where μ_y and σ_y are the mean and standard deviation of RV y.

Then compute the cumulative probabilities realized at these thresholds $\{Y_t\}$, namely

$$P_t = C_y(Y_t) \quad \text{for } 1 \le t \le T \ , \tag{13}$$

and print out M, dB_y, μ_y, σ_y, T, $\{Y_t\}$, $\{\underline{Y}_t\}$, $\{P_t\}$. The false alarm probabilities are $\{1-P_t\}$.

An alternative choice is to space the thresholds $\{Y_t\}$ equally in decibels. That is, defining

$$D_t = 10 \log_{10}(Y_t) \quad \text{for } 1 \le t \le T \tag{14}$$

as the threshold in decibels, we take D_1 as given in terms of Y_1 and we take

$$D_T = D_1 + dB_y \ . \tag{15}$$

The intermediate decibel thresholds are then selected according to the equal spacing rule

$$D_t = D_1 + \frac{D_T - D_1}{T - 1}(t - 1) \quad \text{for } 1 \le t \le T \ . \tag{16}$$

The power thresholds can then be evaluated as

$$Y_t = 10^{D_t/10} \quad \text{for } 1 \le t \le T \ . \tag{17}$$

The cumulative probabilities at these latter power thresholds follow from (13) as before.

Another possibility is to simply set the thresholds according to

$$Y_t = \mu_y + \sigma_y \underline{Y}_t \quad \text{for } 1 \le t \le T \ , \tag{18}$$

where normalized thresholds $\{\underline{Y}_t\}$ are preset constants determined by the user. In this latter case, probability P_1 in (9) would not be realized at initial threshold $Y_1 = \mu_y + \sigma_y \underline{Y}_1$. In any event, the desired printouts are the quantities listed under (13).

Finally, we could set the thresholds such that preset values of probabilities $\{P_t\}$ are realized for all $1 \le t \le T$. That is, solve (13) for the thresholds according to

$$Y_t = \tilde{C}_y(P_t) \quad \text{for } 1 \le t \le T \; . \tag{19}$$

This amounts to setting T different false alarm probabilities $\{1-P_t\}$.

ALTERNATIVE SIZE TRANSFORMATION

Now consider the new RV z obtained by changing the parameter value from M to N in the given nonlinear transformation in (5):

$$z = f(x_0, x_1, \ldots, x_N) \ . \tag{20}$$

N can be larger or smaller than M. Let the CDF of RV z be $C_z(Z)$. We now choose new thresholds, $\{Z_t\}$ for $1 \le t \le T$, such that the probabilities $\{P_t\}$ in (13) are maintained for RV z; that is, we choose thresholds $\{Z_t\}$ for RV z in (20) such that

$$C_z(Z_t) = P_t \ , \quad \text{or } Z_t = \tilde{C}_z(P_t) \quad \text{for } 1 \le t \le T \ . \tag{21}$$

These thresholds $\{Z_t\}$, to be employed for RV z, will not necessarily have equal spacing, as did the thresholds $\{Y_t\}$ in (11), for example, for RV y. The normalized thresholds for RV z are

$$\underline{Z}_t = \frac{Z_t - \mu_z}{\sigma_z} \quad \text{for } 1 \le t \le T \ , \tag{22}$$

where μ_z and σ_z are the mean and standard deviation of RV z. Then print out N, dB_z, μ_z, σ_z, T, $\{Z_t\}$, $\{\underline{Z}_t\}$, $\{P_t\}$, where

$$dB_z \equiv 10 \log_{10}(Z_T/Z_1) \ . \tag{23}$$

DIFFERENT NUMBER OF THRESHOLDS

If RV z is to be subject to a different number, U, of thresholds than RV y was, it is not always reasonable to try to maintain the set of T probabilities $\{P_t\}$ realized in (13). (U can be larger or smaller than T.) One alternative is to maintain the edge probabilities P_1 and P_T according to (21), thereby determining values for Z_1 and Z_T. Then choose a different complete set of thresholds $\{Z_u\}$ for RV'z according to equal spacing rule

$$Z'_u = Z_1 + \frac{Z_T - Z_1}{U - 1}(u - 1) \quad \text{for } 1 \le u \le U \ . \tag{24}$$

We can then evaluate the cumulative probabilities at these latter threshold values as

$$P'_u = C_z(Z'_u) \quad \text{for } 1 \le u \le U \ . \tag{25}$$

Of course, $P'_1 = P_1$ and $P'_U = P_T$, since $Z'_1 = Z_1$ and $Z'_U = Z_T$. This also means that dB_z is still given by (23).

It must be noted that this change in philosophy, namely to maintain only edge probabilities P_1 and P_T, will not reduce to the earlier results when we set U = T here. The new thresholds in z, given by (24), are equally spaced even if U = T, whereas the former thresholds given by (21) are not.

Print out N, dB_z, μ_z, σ_z, U, $\{Z'_u\}$, $\{\underline{Z}'_u\}$, $\{P'_u\}$, where $\{\underline{Z}'_u\}$ are the normalized thresholds

$$\underline{Z}'_u = \frac{Z'_u - \mu_z}{\sigma_z} \quad \text{for } 1 \le u \le U \ . \tag{26}$$

SIGNAL-TO-NOISE RATIO REQUIREMENTS

Let us return to the original nonlinear transformation (5) characterized by parameter M. The CDF of RV y, with signal present in just <u>one</u> of the input RVs $\{x_m\}$, is denoted by $C_y(Y;R)$, where R is the input signal-to-noise ratio (SNR) in that particular x_m RV containing a signal. If we want this new CDF to realize a specified probability value S_1 at all the thresholds $\{Y_t\}$ in (11), the required SNRs $\{R_t\}$ must satisfy

$$S_1 = C_y(Y_t;R_t) \quad \text{for } 1 \le t \le T \; . \tag{27}$$

From (13) and (27), we know that

$$P_t = C_y(Y_t) = C_y(Y_t;0) \quad \text{for } 1 \le t \le T \; . \tag{28}$$

Therefore, specified probability S_1 in (27) must satisfy

$$S_1 \le P_t \quad \text{for } 1 \le t \le T \; , \tag{29}$$

in order that nonnegative SNRs $\{R_t\}$ can result as solutions to (27). That is, each cumulative distribution value must be decreased from P_t to S_1, meaning that each exceedance probability has been increased from false alarm probability value $1-P_t$ to detection probability value $Pd_1 = 1-S_1$. The actual determination of CDF $C_y(Y;R)$ will have to wait until after we have specified and analyzed the nonlinear transformations (5) of interest, for the case of noise-only present.

If several detection probabilities, like $Pd_1 = .5$, $Pd_2 = .9$, $Pd_3 = .99$, are specified, there will be a set of equations like (27) for each case, namely

$$1 - Pd_j = S_j = C_y(Y_t;R_t) \quad \text{for } 1 \le t \le T \;; \quad j = 1,2,3 \; . \qquad (30)$$

STATISTICS OF NORMALIZER

Let RV y be obtained by means of a normalization procedure from IID RVs $\{x_m\}$, $0 \le m \le M$, according to transformation

$$y = \frac{x_0}{\frac{1}{M}\left(x_1 + \cdot \cdot \cdot + x_M\right)} \equiv \frac{x_0}{a} , \tag{31}$$

where we assume that $x_m \ge 0$ for all m. The average RV, a, in the denominator of (31) has, respectively, CF and PDF

$$f_a(\xi) = [f_x(\xi/M)]^M ,$$

$$p_a(u) = \frac{1}{2\pi} \int d\xi \, \exp(-iu\xi) \, [f_x(\xi/M)]^M . \tag{32}$$

The CDF of RV y in (31) is, using the statistical independence of x_0 and a,

$$C_y(Y) = \text{Prob}(y < Y) = \text{Prob}(x_0 < Ya) =$$

$$= \int_0^\infty dt \, p_a(t) \int_0^{Yt} du \, p_x(u) = \int_0^\infty dt \, p_a(t) \, C_x(Yt) . \tag{33}$$

The corresponding PDF of RV y is, upon differentiation,

$$p_y(u) = \int_0^\infty dt \, t \, p_a(t) \, p_x(ut) . \tag{34}$$

The moments oi RV y can be found from a couple of forms:

$$\mu_y(j) = \overline{y^j} = \int_0^\infty du\ u^j\ p_y(u) =$$

$$= \overline{x_0^j}\ \overline{a^{-j}} = \mu_x(j)\ \mu_a(-j) = \int du\ u^j\ p_x(u) \times \int dt\ t^{-j}\ p_a(t)\ . \quad (35)$$

Convergence of the last integral in (35) may not occur for larger values of j; that is, due to the division in (31), RV y may not have finite higher-order moments.

EXAMPLE

The example presented in (4) is utilized here. From (4) and (32), the CF and PDF of average RV a in (31) are

$$f_a(\xi) = \frac{1}{(1 - i\xi/M)^M} \quad (36)$$

and

$$p_a(u) = \frac{M^M\ u^{M-1}\ \exp(-Mu)}{(M-1)!} \quad \text{for } u > 0\ , \quad (37)$$

respectively. The CDF of RV y then follows from (33) and (4) as

$$C_y(Y) = \int_0^\infty dt\ \frac{M^M\ t^{M-1}\ \exp(-Mt)}{(M-1)!}\ [1 - \exp(-Yt)] =$$

$$= 1 - \left(1 + \frac{Y}{M}\right)^{-M} \quad \text{for } Y > 0\ . \quad (38)$$

The PDF of RV y is therefore

$$p_y(u) = \left(1 + \frac{u}{M}\right)^{-M-1} \quad \text{for } u > 0 \ . \tag{39}$$

The inverse function to CDF $C_y(Y)$ in (38) is

$$\tilde{C}_y(P) = M\left[(1 - P)^{-1/M} - 1\right] \quad \text{for } 0 < P < 1 \ . \tag{40}$$

The j-th moment of RV y is given by

$$\mu_y(j) = \overline{y^j} = \int du\ u^j\ p_y(u) = \int_0^\infty du\ u^j \left(1 + \frac{u}{M}\right)^{-M-1} =$$

$$= M^{j+1}\ B(j+1,M-j) = M^{j+1}\ \frac{\Gamma(j+1)\ \Gamma(M-j)}{\Gamma(M+1)} = \frac{j!\ M^j}{(M-j)_j} \quad \text{for } j < M \ , \tag{41}$$

where we used (39) and [1; 3.194 3 and 8.384 1]. In particular,

$$\overline{y} = \frac{M}{M-1} \ , \quad \overline{y^2} = \frac{2M^2}{(M-2)(M-1)} \ , \quad \sigma_y^2 = \frac{M^3}{(M-2)(M-1)^2} \quad \text{for } M > 3 \ . \tag{42}$$

The condition $M > 3$ is necessary for RV y to possess a finite variance. We now have all the quantities required for application in the previous section on threshold resetting.

For $M = \infty$, $f_a(\xi)$ in (36) equals $\exp(i\xi)$; that is, RV a in (31) is equal to the constant 1. This corresponds to no normalization and, in fact, (31) reduces to $y = x_o$. Also, (38) - (41) reduces to (4), as expected.

CHANGE IN SIZE OF NORMALIZER

If we now change from size M to N in normalizer (31), we obtain RV z as considered in (20) and sequel. Its CDF follows, by similarity of form to (38), as

$$C_z(Z) = 1 - \left(1 + \frac{Z}{N}\right)^{-N} \quad \text{for } Z > 0 \ . \qquad (43)$$

Its inverse function is

$$\tilde{C}_z(P) = N\left[(1 - P)^{-1/N} - 1\right] \quad \text{for } 0 < P < 1 \ . \qquad (44)$$

Reference to (42) also reveals that the mean and standard deviation of RV z are

$$\mu_z = \frac{N}{N-1} \ , \qquad \sigma_z = \frac{N}{N-1}\left(\frac{N}{N-2}\right)^{\frac{1}{2}} \quad \text{for } N > 3 \ . \qquad (45)$$

The new thresholds are given by (21) and (22).

STATISTICS OF OR-ING DEVICE

Let $\{x_m\}$, $1 \le m \le M$, be IID RVs with common PDF $p_x(u)$ and CDF $C_x(X)$. The CDF and PDF of the maximum RV

$$y = \max\{x_1, x_2, \ldots, x_M\} , \tag{46}$$

yielded by an or-ing device, are then

$$C_y(Y) = [C_x(Y)]^M , \tag{47}$$

$$p_y(u) = M\,[C_x(u)]^{M-1}\,p_x(u) , \tag{48}$$

respectively. The inverse function to CDF C_y in (47) is

$$\tilde{C}_y(P) = \tilde{C}_x\left(P^{1/M}\right) \quad \text{for } 0 < P < 1 . \tag{49}$$

Here, again, $\tilde{C}_x$ is the inverse function to CDF C_x of RV x_m. The CF of RV y follows from (48) as

$$f_y(\xi) = \int du\,\exp(i\xi u)\,M\,[C_x(u)]^{M-1}\,p_x(u) . \tag{50}$$

The moments of RV y may be obtained from (48) in the form

$$\mu_y(j) = \int du\,u^j\,M\,[C_x(u)]^{M-1}\,p_x(u) . \tag{51}$$

Alternatively, the cumulants can be obtained by a power series expansion of the natural logarithm of CF $f_y(\xi)$ in (50).

EXAMPLE

We again use the results given in (4). Substitution into (50) yields

$$f_y(\xi) = \int_0^\infty du \exp(i\xi u - u)\, M\, [1 - \exp(-u)]^{M-1} =$$

$$= M \int_0^1 dt\, t^{-i\xi}\, (1-t)^{M-1} = \frac{\Gamma(1-i\xi)\, \Gamma(M+1)}{\Gamma(M+1-i\xi)} =$$

$$= \left[\prod_{m=1}^{M}\left(1 - \frac{i\xi}{m}\right)\right]^{-1} , \tag{52}$$

where we let t = exp(-u) and used [1; 8.380 1 and 8.384 1]. This is a very compact form and is easily computed numerically, if necessary. This result illustrates that RV y has the same statistics as RV w defined by

$$w \equiv \sum_{m=0}^{M} \frac{x_m}{m} = x_1 + \frac{1}{2}x_2 + \cdots + \frac{1}{M}x_M . \tag{53}$$

In order to determine the cumulants of RV y, we consider

$$\ln f_y(\xi) = -\sum_{m=1}^{M} \ln\left(1 - \frac{i\xi}{m}\right) = \sum_{m=1}^{M} \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{i\xi}{m}\right)^k . \tag{54}$$

The cumulants follow immediately as

$$\chi_y(j) = (j-1)! \sum_{m=1}^{M} \frac{1}{m^j} \quad \text{for } j \geq 1 \ . \tag{55}$$

In particular, the mean and variance of RV y are

$$\mu_y = \chi_y(1) = \sum_{m=1}^{M} \frac{1}{m} \ , \quad \sigma_y^2 = \chi_y(2) = \sum_{m=1}^{M} \frac{1}{m^2} \ , \tag{56}$$

respectively. It is seen that the mean of RV y increases without limit, in fact, logarithmic with M. On the other hand, the variance saturates at $\pi^2/6$, meaning that the standard deviation σ_y of RV y cannot exceed $\pi/6^{\frac{1}{2}} = 1.283$.

The CDF of y follows from (47) and (4) as

$$C_y(Y) = [1 - \exp(-Y)]^M \quad \text{for } Y > 0 \ , \tag{57}$$

with corresponding inverse function

$$\tilde{C}_y(P) = -\ln\left(1 - P^{1/M}\right) \quad \text{for } 0 < P < 1 \ . \tag{58}$$

For M = 1, there is no or-ing, and (46) reduces to $y = x_1$. Also, (52), (55), (57), and (58) reduce to (4), as expected.

CHANGE IN SIZE OF OR-ING DEVICE

If we change from size M to N in or-ing device (46), we obtain RV z considered in (20) and sequel. Its CDF is, by similarity in form to (57),

$$C_z(Z) = [1 - \exp(-Z)]^N \quad \text{for } Z > 0 . \tag{59}$$

The corresponding inverse function is

$$\tilde{C}_z(P) = -\ln\left(1 - P^{1/N}\right) \quad \text{for } 0 < P < 1 . \tag{60}$$

Reference to (56) reveals that the mean and variance of RV z are given by

$$\mu_z = \sum_{n=1}^{N} \frac{1}{n} , \qquad \sigma_z^2 = \sum_{n=1}^{N} \frac{1}{n^2} . \tag{61}$$

The new thresholds for RV z are given by (21) and (22).

STATISTICS OF NORMALIZER AND OR-ING DEVICE

Here, we consider a set of K(M+1) IID RVs $\{w_{km}\}$ subject to both normalization and or-ing, according to

$$a_k = \frac{1}{M}\sum_{m=1}^{M} w_{km} \ , \qquad x_k = \frac{w_{ko}}{a_k} \quad \text{for } 1 \le k \le K \ , \tag{62}$$

$$y = \max(x_1, x_2, \ldots, x_K) \ . \tag{63}$$

The statistics of the normalization portion, namely $\{x_k\}$ for $1 \le k \le K$, were previously considered in (31) - (35) for the general case, and then specialized to an example in (36) - (42). Also, the analysis of the or-ing portion was conducted in (46) - (51) and then specialized to an example in (52) - (58). We will rely heavily on those results in order to minimize the presentation in this section.

EXAMPLE

We presume that all the input RVs $\{w_{km}\}$ in (62) have the common exponential PDF used in example (4) for all the earlier results here. Then, by reference to (31) and (39), we can state that the common PDF of RVs $\{x_k\}$ defined in (62) is

$$p_x(u) = \left(1 + \frac{u}{M}\right)^{-M-1} \quad \text{for } u > 0 \ . \tag{64}$$

Similarly, the CDF of RVs $\{x_k\}$ follows, by reference to (38), as

$$C_x(X) = 1 - \left(1 + \frac{X}{M}\right)^{-M} \quad \text{for } X > 0 . \tag{65}$$

The CDF of RV y defined in (63) is then

$$C_y(Y) = [C_x(Y)]^K = \left[1 - \left(1 + \frac{Y}{M}\right)^{-M}\right]^K \quad \text{for } Y > 0 . \tag{66}$$

Its inverse function follows readily as

$$\tilde{C}_y(P) = M\left[\left(1 - P^{1/K}\right)^{-1/M} - 1\right] \quad \text{for } 0 < P < 1 . \tag{67}$$

The PDF of RV y can be obtained by differentiation of (66):

$$p_y(u) = K\,[C_x(u)]^{K-1}\, p_x(u) =$$

$$= K\left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^{K-1} \left(1 + \frac{u}{M}\right)^{-M-1} \quad \text{for } u > 0 . \tag{68}$$

The j-th moment of RV y can be found from the integral

$$\mu_y(j) = \int_0^\infty du\; u^j\, p_y(u) \quad \text{for } j < M . \tag{69}$$

where we have taken advantage of the fact that RV y can never be negative. However, a useful alternative in some cases is afforded by employing integration by parts on (69), with the result, for $j \ge 1$, that

$$\mu_y(j) = j \int_0^{\infty} du\, u^{j-1}\, [1 - C_y(u)]\ , \tag{70}$$

where we assume that

$$\lim_{u\to+\infty} u^j [1 - C_y(u)] = 0\ . \tag{71}$$

This requirement is tantamount to presuming that the j-th moment $\mu_y(j)$ exists, that is, $j < M$.

When we use CDF (66) for RV y, then j-th moment (70) becomes

$$\mu_y(j) = j \int_0^{\infty} du\, u^{j-1} \left\{1 - \left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^K\right\} . \tag{72}$$

In order to evaluate this integral, let, for the moment,

$$Q = \left(1 + \frac{u}{M}\right)^{-M} . \tag{73}$$

Then, the bracketed term in (72) can be expanded according to

$$[1 - Q]^K = \sum_{k=0}^{K} \binom{K}{k} (-Q)^k = 1 + \sum_{k=1}^{K} (-1)^k \binom{K}{k} \left(1 + \frac{u}{M}\right)^{-Mk} . \tag{74}$$

Employment of this result in (72) yields

$$\mu_y(j) = j \int_0^\infty du\ u^{j-1} \sum_{k=1}^{K} (-1)^{k-1} \binom{K}{k} \left(1 + \frac{u}{M}\right)^{-Mk} =$$

$$= j \sum_{k=1}^{K} (-1)^{k-1} \binom{K}{k} \int_0^\infty du\ u^{j-1} \left(1 + \frac{u}{M}\right)^{-Mk} =$$

$$= j!\ M^j \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k-1}}{(Mk-j)_j} \quad \text{for } 1 \le j < M\ , \tag{75}$$

where we employed [1; 3.194 3 and 8.384 1] and simplified the end result. For $K = 1$, (75) reduces to (41), as it must.

The first two moments for RV y follow readily from (75) as

$$\mu_y(1) = M \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k-1}}{Mk-1} \tag{76}$$

and

$$\mu_y(2) = 2M^2 \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k-1}}{(Mk-2)(Mk-1)} \quad \text{for } M > 2\ . \tag{77}$$

Both of these results are easily computed by means of recurrences; however, a bad feature of these two sums is that they are alternating series and lose significance for large K due to the binomial coefficient which gets very large. A backup procedure is to revert to numerical integration of (68) - (69) or (70) - (72), both of which integrands can never go negative and which decay as u^{j-1-M} for large u.

However, better alternatives to the first and second moments have been found, that are not subject to cancellation and loss of significance. Namely, it is shown in appendix A that

$$\mu_y(1) = M[F_1 - 1] , \quad \sigma_y = M\left(F_2 - F_1^2\right)^{\frac{1}{2}} , \tag{78}$$

where

$$F_1 = \prod_{k=1}^{K} \left\{ \frac{k}{k - \frac{1}{M}} \right\} , \quad F_2 = \prod_{k=1}^{K} \left\{ \frac{k}{k - \frac{2}{M}} \right\} \text{ for } M > 2 . \tag{79}$$

These finite products are very useful and retain significance even for large K, where (76) and (77) are useless. The very compact BASIC program listed below computes both quantities in (78). The program has been written so as to avoid overflow, even when K is large.

```
K=
M=                ! M > 2
A=1/M
B=2/M
F1=F2=1
FOR Ks=1 TO K
F1=F1*Ks/(Ks-A)
F2=F2*Ks/(Ks-B)
NEXT Ks
Muy=M*(F1-1)
Sigy=M*SQR(F2-F1*F1)
```

(80)

By using the techniques in appendix A and laboriously expanding out (75) for j = 3, it has been found that

$$\mu_y(3) = M^3(F_3 - 3F_2 + 3F_1 - 1) ,$$

where we define products

$$F_m = \prod_{k=1}^{K} \left\{\frac{k}{k - \frac{m}{M}}\right\} \quad \text{for } 0 \le m < M .$$

Notice that $F_o = 1$. Based on this result and (78), we conjectured that the j-th moment is generally given by

$$\mu_y(j) = M^j \sum_{n=0}^{j} (-1)^n \binom{j}{n} F_{j-n} \quad \text{for } j < M .$$

In fact, this has been confirmed numerically for several values of M, K, and j. For large j, the alternating character of this series would also suffer from loss of significance; however, for the low order moments of general interest, this is not a problem.

The third and fourth cumulants of RV y were also computed in terms of sequence $\{F_n\}$; they turned out to be

$$\chi_y(3) = M^3 \left[F_3 - 3F_2F_1 + 2F_1^3\right] ,$$

$$\chi_y(4) = M^4 \left[F_4 - 4F_3F_1 - 3F_2^2 + 12F_2F_1^2 - 6F_1^4\right] .$$

But these rules for obtaining cumulants $\{\chi_y(j)\}$ from products $\{F_n\}$ are <u>identical</u> to the general rules for obtaining cumulants from moments, within the factor M^j; see, for example, [4; page 70, (3.41)]. Thus, we have a very efficient and accurate method for obtaining the low-order cumulants directly from the finite products $\{F_n\}$. The case for $\chi_y(1)$ in (78) differs slightly from the usual rule, in the need to subtract 1.

ANALYTICAL CHECKS

Numerous checks on the results above are possible. When $M = \infty$ (no normalization), the CDF of y in (66) reduces to (57), where it must be remembered that K in this section on combined normalization and or-ing corresponds to M in the section on or-ing alone. On the other hand, if $K = 1$ (no or-ing) in (66), the result in (38) correctly emerges.

With regard to the inverse CDF in (67), it reduces to (58) for $M = \infty$, whereas it reduces to (40) for $K = 1$. The PDF of RV y, given in (68), reduces to the derivative of (57) for $M = \infty$, whereas it reduces to (39) for $K = 1$. Finally, the first two moments in (78) - (79) reduce, after some manipulations, to (56) for $M = \infty$; on the other hand, the j-th moment for $K = 1$ is best handled from form (75) which correctly reduces to (41).

EXTENSIONS

The case where the normalizer and or-ing device are followed by an averager is discussed in the summary below, and the method of determining the performance of that system is outlined. Another alternative with practical merit is that of normalization followed by averaging and or-ing. Since the CF of the normalizer output is available by a Fourier transform of (39), it can be raised to a power to find the CF at the averager output. Then another Fourier transform can yield the CDF. At this point, we could utilize (47) to find the CDF of the system output.

CHANGES IN SIZES OF NORMALIZER AND OR-ING DEVICE

We now address the change in size of the normalizer in (62) from size M to N and the change in size of the or-ing device in (63) from K to L. There are no restrictions on the relative sizes of the parameters. The new equations are

$$b_k = \frac{1}{N}\sum_{n=1}^{N} w_{kn} \;, \quad y_k = \frac{w_{ko}}{b_k} \quad \text{for } 1 \le k \le L \;, \tag{81}$$

$$z = \max(y_1, y_2, \ldots, y_L) \; . \tag{82}$$

The CDF of RV z follows, by similarity of form to (66), as

$$C_z(Z) = \left[1 - \left(1 + \frac{Z}{N}\right)^{-N}\right]^L \quad \text{for } Z > 0 \; . \tag{83}$$

The inverse function is easily shown to be

$$\tilde{C}_z(P) = N\left[\left(1 - P^{1/L}\right)^{-1/N} - 1\right] \quad \text{for } 0 < P < 1 \; . \tag{84}$$

The first two moments of RV z in (82) are given, by comparison with (78) and (79), as

$$\mu_z(1) = N[G_1 - 1] \;, \quad \sigma_z = N\left(G_2 - G_1^2\right)^{\frac{1}{2}} \;, \tag{85}$$

where

$$G_1 = \prod_{k=1}^{L}\left\{\frac{k}{k - \frac{1}{N}}\right\} \;, \quad G_2 = \prod_{k=1}^{L}\left\{\frac{k}{k - \frac{2}{N}}\right\} \; . \tag{86}$$

INPUT SIGNAL-TO-NOISE RATIO REQUIREMENTS

In this section, we will determine the CDFs of the outputs of the three nonlinear systems considered above, namely (31), (46), and (62) - (63), for the case where a signal is present in <u>one</u> of the input RVs. This will enable us to use the results given in (27) - (29) for determination of required input SNRs for a specified system detection probability $Pd_1 = 1-S_1$.

NORMALIZER

The nonlinear transformation of immediate interest is the normalizer given by (31). The PDF of denominator average RV a is given by (37), while the PDF of numerator RV x_0 with signal present will be modified from (4) to the form

$$p_x(u;R) = \frac{1}{1+R} \exp\left(\frac{-u}{1+R}\right) \quad \text{for } u > 0 , \tag{87}$$

where R is the input power SNR. The corresponding CDF is

$$C_x(X;R) = 1 - \exp\left(\frac{-X}{1+R}\right) \quad \text{for } X > 0 . \tag{88}$$

By reference to (33), (38), and (88), we find the CDF of RV y in (31) for signal present to be

$$C_y(Y;R) = \int_0^\infty dt \, \frac{M^M \, t^{M-1} \exp(-Mt)}{(M-1)!} \left[1 - \exp\left(\frac{-Yt}{1+R}\right)\right] =$$

$$= 1 - \left(1 + \frac{Y/M}{1+R}\right)^{-M} \quad \text{for } Y > 0 . \tag{89}$$

We now employ this result in (27) to obtain

$$S_1 = 1 - \left(1 + \frac{Y_t/M}{1+R_t}\right)^{-M} \quad \text{for } 1 \le t \le T . \tag{90}$$

The solution for the required input SNR R_t is then given by

$$R_t = \frac{Y_t/M}{(1 - S_1)^{-1/M} - 1} - 1 \quad \text{for } 1 \le t \le T . \tag{91}$$

We must repeat here the caution mentioned in (29), namely that specified probability S_1 at threshold Y_t must be less than or equal to probabilities $\{P_t\}$ in order that nonnegative SNRs $\{R_t\}$ result in (91). This is reasonable since it corresponds physically to demanding a larger detection probability when signal is present than when absent.

OR-ING DEVICE

Here, we are interested in the or-ing device characterized by (46) when signal is present in one of the RVs $\{x_m\}$. The CDF of RV y is then given by

$$C_y(Y;R) = C_x(Y;R)\,[C_x(Y)]^{M-1} =$$

$$= \left[1 - \exp\left(\frac{-Y}{1+R}\right)\right]\,[1 - \exp(-Y)]^{M-1} \quad \text{for } Y > 0 , \tag{92}$$

where we used (88) and (4).

If we now employ (92) in (27), we have to satisfy

$$S_1 = \left[1 - \exp\left(\frac{-Y_t}{1+R_t}\right)\right]\,\left[1 - \exp\left(-Y_t\right)\right]^{M-1} \quad \text{for } 1 \le t \le T . \tag{93}$$

The solution for the required input SNR is given by

$$R_t = \frac{Y_t}{-\ln(1 - S_1/Q_t)} - 1 \quad \text{for } 1 \le t \le T , \tag{94}$$

where

$$Q_t = \left[1 - \exp\left(-Y_t\right)\right]^{M-1} \quad \text{for } 1 \le t \le T . \tag{95}$$

Again, (29) must be satisfied.

NORMALIZER AND OR-ING DEVICE

The nonlinear transformation to be investigated here is the combination of normalization and or-ing, as characterized by (62) and (63), when signal is present only in RV w_{10}. Therefore, only RV x_1 in (63) contains signal.

The CDFs of RVs $\{x_k\}$, for $2 \le k \le K$, are given by (65). On the other hand, the CDF for x_1 is available by reference to (89), in the form

$$C_x(X;R) = 1 - \left(1 + \frac{X/M}{1+R}\right)^{-M} \quad \text{for } X > 0 \ . \tag{96}$$

Therefore, the CDF for RV y in (63) is given by

$$C_y(Y;R) = C_x(Y;R) \ [C_x(Y)]^{K-1} =$$

$$= \left[1 - \left(1 + \frac{Y/M}{1+R}\right)^{-M}\right] \left[1 - \left(1 + \frac{Y}{M}\right)^{-M}\right]^{K-1} \quad \text{for } Y > 0 \ , \tag{97}$$

where we used (96) and (65).

When we equate this result to the specified probability S_1 according to (27), we obtain

$$S_1 = \left[1 - \left(1 + \frac{Y_t/M}{1+R_t}\right)^{-M}\right] \left[1 - \left(1 + \frac{Y_t}{M}\right)^{-M}\right]^{K-1} \quad \text{for } 1 \le t \le T \ . \tag{98}$$

The solution for the required input SNRs $\{R_t\}$ is given by

$$R_t = \frac{Y_t/M}{(1 - S_1/Q_t)^{-1/M} - 1} - 1 \quad \text{for } 1 \le t \le T \,, \tag{99}$$

where

$$Q_t = \left[1 - \left(1 + \frac{Y_t}{M}\right)^{-M}\right]^{K-1} \quad \text{for } 1 \le t \le T \,. \tag{100}$$

Restriction (29) must be satisfied here also.

Finally, if several detection probabilities Pd_1, Pd_2, Pd_3, are of interest, we must satisfy (30), where

$$\max\{1-P_t\} \le \min\{Pd_j\} \,. \tag{101}$$

As checks on the results in this subsection on combined normalization and or-ing, we observe for $M = \infty$ (that is, no normalization), (100) reduces to (95), where K here corresponds to M there for or-ing alone. Also, (99) reduces to (94). On the other hand, for finite M, but with $K = 1$ (that is, no or-ing), then (100) reduces to $Q_t = 1$, in which case (99) reduces to (91).

SUMMARY

We have determined the false alarm and detection probabilities for three different nonlinear transformations, namely a normalizer, an or-ing device, and a normalizer followed by or-ing. In particular, results are given for the following statistics of the outputs of each device: the PDF, the CDF, the inverse CDF, and either the moments or the cumulants (depending on their relative tractability). These results are sufficient to compute the receiver operating characteristics (ROCs) of the processors, if desired. However, we have also been able to solve explicitly for the input SNR required to realize specified false alarm and detection probabilities; this largely circumvents the need to compute ROCs.

Two programs, with numerical examples of their execution, are listed in appendix B. The first corresponds to the case where the number T of thresholds is kept fixed as the size of the nonlinear transformation is changed from M to N; see (20) - (23). On the other hand, the second program allows the number of thresholds to change from T to U as the size of the nonlinear transformation is changed from M to N; see (24) - (26).

Both programs are written for the general case where there is both normalization (of size M) and or-ing (of size K) included in the data processing; see (62) - (63). By making M infinite, the program will handle the case of or-ing alone; on the other hand, by setting K = 1, the program addresses the case of normalization alone. Thus, these two programs cover all the cases addressed in

this investigation. (Since it is not possible to actually set M infinite in a program, this situation is handled by setting M to any value less than or equal to 2, in order to flag this case in the program, and then branching appropriately at various points in the program. The substitute equations for this case of infinite M come, of course, from the earlier analysis for or-ing alone. For finite variance, normalization requires $M > 2$; see, for example, (77) or (79).)

We have not included the effects of averaging after the normalization and/or or-ing in this study. Hence, the required input SNRs calculated here sometimes turn out to be rather large. The exact analysis including averaging would be rather involved, since the new decision variable would have a characteristic function given by a power of the characteristic function $f_y(\xi)$ of current output variable y; that is, from (68),

$$f_y(\xi) = K \int_0^{\infty} du \exp(i\xi u) \left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^{K-1} \left(1 + \frac{u}{M}\right)^{-M-1} . \quad (102)$$

This is probably best handled through the use of fast Fourier transforms. The integrand decays as u^{-M-1} for large u, which is attractive since M, the normalization size, is generally fairly large in order to guarantee decent performance.

In the meantime, in order to get a rather rough idea of the improvement attainable by employment of averaging, it is suggested that the rule of thumb [3; (C-10)] for the input signal-to-noise ratio improvement in dB, $-5 \log A$, be used, where A is the number of independent quantities averaged.

APPENDIX A. SIMPLIFICATION OF SUMS (76) AND (77)

Here, we will convert the alternating series in (76) and (77) into finite products that retain significance, even for large values of K. We begin with the first line of (52) and expand the power term in a binomial series, obtaining

$$f_y(\xi) = M \int_0^\infty du \, \exp(i\xi u - u) \sum_{k=0}^{M-1} (-1)^k \binom{M-1}{k} \exp(-ku) =$$

$$= M \sum_{k=0}^{M-1} \binom{M-1}{k} \frac{(-1)^k}{k + 1 - i\xi} . \qquad \text{(A-1)}$$

Now, equate (A-1) to its alternative expression in the last line of (52), and then replace M everywhere by K+1, getting

$$(K+1) \sum_{k=0}^{K} \binom{K}{k} \frac{(-1)^k}{k + 1 - i\xi} = \prod_{k=1}^{K+1} \left(\frac{k}{k - i\xi}\right) . \qquad \text{(A-2)}$$

Now let $z = 1 - i\xi$ in (A-2) and simplify; there follows

$$\sum_{k=0}^{K} \binom{K}{k} \frac{(-1)^k}{k + z} = \frac{K!}{(z)_{K+1}} . \qquad \text{(A-3)}$$

Thus, the alternating series has been converted into a finite product which is useful for all values of K without loss of significance.

The use of (A-3) on (76) yields the result quoted in (78) and (79) for the first moment $\mu_y(1)$. On the other hand, in order to convert (77), it is necessary to take the preliminary step of breaking the rational function into two parts according to

$$\frac{1}{(k-b)(k-a)} = \frac{1}{b-a}\left[\frac{1}{k-b} - \frac{1}{k-a}\right], \tag{A-4}$$

and then to use (A-3) once with $z = -1/M$ and once with $z = -2/M$. After manipulation, simplification, and cancellation of common terms with the square of $\mu_y(1)$, the end result for the standard deviation of RV y is found to be just the second term in (78). The results in (78) - (79) have been numerically checked with the original defining integral (72), for $j = 1$ and $j = 2$, for several values of M and K; they agree within the accuracy of the computer employed.

A more general result than (A-3) is available by means of a different approach. First, for general values of a, note that

$$\binom{a}{k} = \frac{(-1)^k \, (-a)_k}{k!}, \qquad \frac{1}{k+z} = \frac{1}{z}\,\frac{(z)_k}{(z+1)_k}. \tag{A-5}$$

Then, the following alternating sum can be manipulated into a more useful form, namely

$$\sum_{k=0}^{\infty} \binom{a}{k} \frac{(-1)^k}{k+z} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{(-a)_k \, (z)_k}{(z+1)_k \, k!} = \frac{1}{z}\, F(-a,z;z+1;1) = \frac{\Gamma(1+a)\,\Gamma(z)}{\Gamma(1+a+z)}, \tag{A-6}$$

where we used [2; 15.1.1 and 15.1.20]. If we set $a = K$ in (A-6), it reduces to (A-3). (A-6) is also equal to $B(1+a,z)$.

APPENDIX B. PROGRAMS

There are two programs listed in this appendix, both in BASIC for the Hewlett-Packard 9000 computer. None of the variables are declared INTEGER; thus, for example, input parameters M, K, N, L are all treated as REAL variables throughout.

The first program, listed on pages 42 - 44, requires that the number of thresholds T be maintained the same when the sizes, M and K, of the normalizer and or-ing device, respectively, are changed to N and L. On the other hand, the second program, listed on pages 46 - 49, allows the number of thresholds to change from T to U, which can be either larger or smaller.

The listings are heavily keyed to the explicit equations in the main text; this should enable the user to identify and modify particular manipulations if desired. It should be noted that, in the programs, the or-ing size begins at value K and gets changed to L. If these results are to be compared with the or-ing only results in the text, where the parameter M was used, it is necessary to make the switch from K in the program to M in the text. Example outputs from both programs are presented after the listing for each case.

```
10  ! NORMALIZER & ORING, EQUISPACED IN POWER, SAME NO. OF THRESHOLDS
20    P1=.85                    !  SPECIFIED PROBABILITY, (9)
30    Dby=10                    !  DECIBEL RATIO OF THRESHOLDS, (10)
40    T=7                       !  NUMBER OF THRESHOLDS, (10)
50    M=8                       !  INITIAL NORMALIZER SIZE > 2, (62)
60  ! FOR NO INITIAL NORMALIZATION, THAT IS, M INFINITE, SET M <= 2
70    K=12                      !  INITIAL OR-ING SIZE > 0, (63)
80    N=16                      !  NEW NORMALIZER SIZE > 2, (81)
90  ! FOR NO NEW NORMALIZATION, THAT IS, N INFINITE, SET N <= 2
100   L=24                      !  NEW OR-ING SIZE > 0, (82)
110   Pd1=.5                    !  SPECIFIED
120   Pd2=.9                    !   DETECTION
130   Pd3=.99                   !    PROBABILITIES, (30)
140   PRINT "P1 =";P1;"      Dby =";Dby;"       T =";T
150   PRINT "M =";M;"         K =";K;"         N =";N;"          L =";L
160   REDIM Y(1:T),Yb(1:T),P(1:T),Z(1:T),Zb(1:T)
170   DIM Y(99),Yb(99),P(99),Z(99),Zb(99)
180   Rk=1/K                    !  (67)
190   Rl=1/L                    !  (84)
200   IF M>2 THEN 300
210   F1=F2=0                   !  M INFINITE
220   FOR Ms=1 TO K             !  (56), m
230   R1=1/Ms
240   F1=F1+R1
250   F2=F2+R1*R1
260   NEXT Ms
270   Muy=F1                    !  (56)
280   Sigy=SQR(F2)              !  (56)
290   GOTO 390
300   R1=1/M                    !  M > 2
310   R2=2/M
320   F1=F2=1
330   FOR Ks=1 TO K             !  (79), k
340   F1=F1*Ks/(Ks-R1)          !  (79)
350   F2=F2*Ks/(Ks-R2)          !  (79)
360   NEXT Ks
370   Muy=M*(F1-1)              !  (78)
380   Sigy=M*SQR(F2-F1*F1)      !  (78)
390   IF N>2 THEN 490
400   G1=G2=0                   !  N INFINITE
410   FOR Ns=1 TO L             !  (61), n
420   R1=1/Ns
430   G1=G1+R1
440   G2=G2+R1*R1
450   NEXT Ns
460   Muz=G1                    !  (61)
470   Sigz=SQR(G2)              !  (61)
480   GOTO 580
490   R1=1/N                    !  N > 2
500   R2=2/N
510   G1=G2=1
520   FOR Ks=1 TO L             !  (86), k
530   G1=G1*Ks/(Ks-R1)          !  (86)
540   G2=G2*Ks/(Ks-R2)          !  (86)
550   NEXT Ks
560   Muz=N*(G1-1)              !  (85)
570   Sigz=N*SQR(G2-G1*G1)      !  (85)
580   PRINT "Muy =";Muy;"          Sigy =";Sigy
590   PRINT "Muz =";Muz;"          Sigz =";Sigz
600   PRINT
```

```
610  ! EQUISPACED IN DECIBELS: REMOVE 880-970 AND INSERT 620-710
620  ! R=1-P1^Rk                 !  (9) & (67)
630  ! IF M>2 THEN 660
640  ! Y1=-LOG(R)                !  (58)
650  ! GOTO 670
660  ! Y1=M*(R^(-1/M)-1)         !  (57)
670  ! D1=10*LGT(Y1)             !  (14)
680  ! Deld=Dby/(T-1)            !  (15) & (16)
690  ! FOR Ts=1 TO T             !  (16), t
700  ! Dt=D1+Deld*(Ts-1)         !  (16)
710  ! Y(Ts)=Y=10^(Dt/10)        !  (17)
720  ! PRESET CONSTANTS: REMOVE 880-980 AND INSERT 730-760
730  ! DATA 1,3,5,7,9,10,11      !  USER MUST INPUT T NUMBERS
740  ! READ Yb(*)                !  NORMALIZED Y THRESHOLDS
750  ! FOR Ts=1 TO T             !  (18), t
760  ! Y(Ts)=Y=Muy+Sigy*Yb(Ts)!  (18)
770  ! PRESET PROBABILITIES: REMOVE 880-1030 AND INSERT 780-870
780  ! DATA .9,.99,.999,.9999,.99999,.999999,.9999999
790  ! READ P(*)                 !  (19)
800  ! FOR Ts=1 TO T             !  (19), t
810  ! P=P(Ts)
820  ! R=1-P^Rk                  !  (58)
830  ! IF M>2 THEN 860
840  ! Y(Ts)=Y=-LOG(R)           !  (58)
850  ! GOTO 870
860  ! Y(Ts)=Y=M*(R^(-1/M)-1) !  (9) & (67)
870  ! Yb(Ts)=(Y-Muy)/Sigy       !  (12)
880    R=1-P1^Rk                 !  (9) & (67)
890    IF M>2 THEN 920
900    Y1=-LOG(R)                !  (58)
910    GOTO 930
920    Y1=M*(R^(-1/M)-1)         !  (67)
930    Ay=10^(Dby/10)            !  (10)
940    Ytc=Y1*Ay                 !  (10)
950    Dely=(Ytc-Y1)/(T-1)       !  (11)
960    FOR Ts=1 TO T             !  (11), t
970    Y(Ts)=Y=Y1+Dely*(Ts-1) !  (11)
980    Yb(Ts)=(Y-Muy)/Sigy       !  (12)
990    IF M>2 THEN 1020
1000   R=EXP(-Y)                 !  (57)
1010   GOTO 1030
1020   R=(1+Y/M)^(-M)            !  (13) & (66)
1030   P(Ts)=P=(1-R)^K           !  (66)
1040   Q=1-P^Rl                  !  (84)
1050   IF N>2 THEN 1080
1060   Z(Ts)=Z=-LOG(Q)           !  (21) & (60)
1070   GOTO 1090
1080   Z(Ts)=Z=N*(Q^(-1/N)-1) !  (21) & (84)
1090   Zb(Ts)=(Z-Muz)/Sigz       !  (22)
1100   NEXT Ts
1110   Dbz=10*LGT(Z(T)/Z(1))     !  (23)
1120   PRINT " Y THRESHOLDS            NORMALIZED           PROBABILITIES"
1130   FOR Ts=1 TO T
1140   PRINT Y(Ts),Yb(Ts),P(Ts)
1150   NEXT Ts
1160   PRINT
1170   PRINT " Z THRESHOLDS            NORMALIZED           PROBABILITIES"
1180   FOR Ts=1 TO T
1190   PRINT Z(Ts),Zb(Ts),P(Ts)
1200   NEXT Ts
1210   PRINT
1220   PRINT "Dbz =";Dbz
```

```
1230    PRINT
1240    PRINT " Pd1 =";Pd1;"             Pd2 =";Pd2;"             Pd3 =";Pd3
1250    PRINT
1260    PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:"
1270    IF M>2 THEN 1400
1280    FOR Ts=1 TO T             !  (94), t
1290    Y=Y(Ts)
1300    Q=(1-EXP(-Y))^(K-1)       !  (95)
1310    D1=-LOG(1-(1-Pd1)/Q)      !  (94)
1320    Rt1=Y/D1-1                !  (94)
1330    D2=-LOG(1-(1-Pd2)/Q)
1340    Rt2=Y/D2-1
1350    D3=-LOG(1-(1-Pd3)/Q)
1360    Rt3=Y/D3-1
1370    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1380    NEXT Ts
1390    GOTO 1530
1400    Rm=-1/M                   !  (99)
1410    FOR Ts=1 TO T             !  (99), t
1420    Ym=Y(Ts)/M                !  (99) & (100)
1430    Q=(1+Ym)^(-M)             !  (100)
1440    Q=(1-Q)^(K-1)             !  (100)
1450    D1=(1-(1-Pd1)/Q)^Rm-1     !  (99)
1460    Rt1=Ym/D1-1               !  (99)
1470    D2=(1-(1-Pd2)/Q)^Rm-1
1480    Rt2=Ym/D2-1
1490    D3=(1-(1-Pd3)/Q)^Rm-1
1500    Rt3=Ym/D3-1
1510    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1520    NEXT Ts
1530    PRINT
1540    PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:"
1550    IF N>2 THEN 1680
1560    FOR Ts=1 TO T             !  SIMILAR
1570    Z=Z(Ts)                   !  TO
1580    Q=(1-EXP(-Z))^(L-1)       !  (94)
1590    D1=-LOG(1-(1-Pd1)/Q)      !  &
1600    Rt1=Z/D1-1                !  (95)
1610    D2=-LOG(1-(1-Pd2)/Q)
1620    Rt2=Z/D2-1
1630    D3=-LOG(1-(1-Pd3)/Q)
1640    Rt3=Z/D3-1
1650    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1660    NEXT Ts
1670    GOTO 1810
1680    Rn=-1/N                   !  SIMILAR
1690    FOR Ts=1 TO T             !  TO
1700    Zn=Z(Ts)/N                !  (99)
1710    Q=(1+Zn)^(-N)             !  &
1720    Q=(1-Q)^(L-1)             !  (100)
1730    D1=(1-(1-Pd1)/Q)^Rn-1
1740    Rt1=Zn/D1-1
1750    D2=(1-(1-Pd2)/Q)^Rn-1
1760    Rt2=Zn/D2-1
1770    D3=(1-(1-Pd3)/Q)^Rn-1
1780    Rt3=Zn/D3-1
1790    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1800    NEXT Ts
1810    PRINT
1820    END
```

Pl = .85	Dby = 10	T = 7	
M = 8	K = 12	N = 16	L = 24
Muy = 3.94631506068		Sigy = 2.09538979024	
Muz = 4.32438211761		Sigz = 1.69328544323	

Y THRESHOLDS	NORMALIZED	PROBABILITIES
5.7085601983	.841010653879	.85
14.2714004957	4.92752493268	.996679075641
22.8342407932	9.01403921147	.999753631023
31.3970810906	13.1005534903	.999965311792
39.9599213881	17.1870677691	.99999280763
48.5227616855	21.2735820479	.999998067526
57.085601983	25.3600963267	.999999374796

Z THRESHOLDS	NORMALIZED	PROBABILITIES
5.66721821227	.911149446675	.85
11.8779764405	4.46091021044	.996679075641
16.8024176865	7.35912705339	.999753631023
21.0784326281	9.89440414638	.999965311792
24.909916314	12.1571553566	.99999280763
28.4120734509	14.2254168838	.999998067526
31.6575625522	16.1420985125	.999999374796

Dbz = 7.32045232969

Pd1 = .5	Pd2 = .9	Pd3 = .99

SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:

7.18029480646	16.5239686654	26.8808804974
12.6997498763	21.2426978346	31.503743108
14.8466399857	23.3091924052	33.5584316626
16.269595645	24.6986419313	34.9428284925
17.3390310658	25.7492651765	35.9905902914
18.1963923488	26.5945353622	36.8340131363
18.9120679996	27.3017783991	37.5399641554

SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:

7.40601673703	16.6306689407	26.9721139815
11.9559811212	20.4527339701	30.7060466017
13.5624054665	21.9841310487	32.2269170012
14.587035072	22.975309045	33.2129962117
15.3357177535	23.704135924	33.9387696157
15.922698397	24.2777465869	34.5103143719
16.4038217386	24.749160188	34.9802228898

```
10   ! NORMALIZER & ORING, EQUISPACED IN POWER, DIFF. NO. OF THRESHOLDS
20     P1=.85                        !  SPECIFIED PROBABILITY, (9)
30     Dby=10                        !  DECIBEL RATIO OF THRESHOLDS, (10)
40     T=7                           !  INITIAL NUMBER OF THRESHOLDS, (10)
50     U=15                          !  NEW NUMBER OF THRESHOLDS, (24)
60     M=8                           !  INITIAL NORMALIZER SIZE > 2, (62)
70   ! FOR NO INITIAL NORMALIZATION, THAT IS, M INFINITE, SET M <= 2
80     K=12                          !  INITIAL OR-ING SIZE > 0, (63)
90     N=16                          !  NEW NORMALIZER SIZE > 2, (81)
100  ! FOR NO NEW NORMALIZATION, THAT IS, N INFINITE, SET N <= 2
110    L=24                          !  NEW OR-ING SIZE > 0, (82)
120    Pd1=.5                        !  SPECIFIED
130    Pd2=.9                        !   DETECTION
140    Pd3=.99                       !    PROBABILITIES, (30)
150    PRINT "P1 =";P1;"      Dby =";Dby;"       T =";T;"           U =";U
160    PRINT "M =";M;"        K =";K;"          N =";N;"           L =";L
170    REDIM Y(1:T),Yb(1:T),P(1:T),Zp(1:U),Zbp(1:U),Pp(1:U)
180    DIM Y(99),Yb(99),P(99),Zp(99),Zbp(99),Pp(99)
190    Rk=1/K                        !  (67)
200    Rl=1/L                        !  (84)
210    IF M>2 THEN 310
220    F1=F2=0                       !  M INFINITE
230    FOR Ms=1 TO K                 !  (56), m
240    R1=1/Ms
250    F1=F1+R1
260    F2=F2+R1*R1
270    NEXT Ms
280    Muy=F1                        !  (56)
290    Sigy=SQR(F2)                  !  (56)
300    GOTO 400
310    R1=1/M                        !  M > 2
320    R2=2/M
330    F1=F2=1
340    FOR Ks=1 TO K                 !  (79), k
350    F1=F1*Ks/(Ks-R1)              !  (79)
360    F2=F2*Ks/(Ks-R2)              !  (79)
370    NEXT Ks
380    Muy=M*(F1-1)                  !  (78)
390    Sigy=M*SQR(F2-F1*F1)          !  (78)
400    IF N>2 THEN 500
410    G1=G2=0                       !  N INFINITE
420    FOR Ns=1 TO L                 !  (61), n
430    R1=1/Ns
440    G1=G1+R1
450    G2=G2+R1*R1
460    NEXT Ns
470    Muz=G1                        !  (61)
480    Sigz=SQR(G2)                  !  (61)
490    GOTO 590
500    R1=1/N                        !  N > 2
510    R2=2/N
520    G1=G2=1
530    FOR Ks=1 TO L                 !  (86), k
540    G1=G1*Ks/(Ks-R1)              !  (86)
550    G2=G2*Ks/(Ks-R2)              !  (86)
560    NEXT Ks
570    Muz=N*(G1-1)                  !  (85)
580    Sigz=N*SQR(G2-G1*G1)          !  (85)
590    PRINT "Muy =";Muy;"             Sigy =";Sigy
600    PRINT "Muz =";Muz;"             Sigz =";Sigz
610    PRINT
```

```
620  ! EQUISPACED IN DECIBELS: REMOVE 890-980 AND INSERT 630-720
630  ! R=1-P1^Rk                  !  (9) & (67)
640  ! IF M>2 THEN 670
650  ! Y1=-LOG(R)                 !  (58)
660  ! GOTO 680
670  ! Y1=M*(R^(-1/M)-1)          !  (67)
680  ! D1=10*LGT(Y1)              !  (14)
690  ! Deld=Dby/(T-1)             !  (15) & (16)
700  ! FOR Ts=1 TO T              !  (16), t
710  ! Dt=D1+Deld*(Ts-1)          !  (16)
720  ! Y(Ts)=Y=10^(Dt/10)         !  (17)
730  ! PRESET CONSTANTS: REMOVE 890-990 AND INSERT 740-770
740  ! DATA 1,3,5,7,9,10,11       !  USER MUST INPUT T NUMBERS
750  ! READ Yb(*)                 !  NORMALIZED Y THRESHOLDS
760  ! FOR Ts=1 TO T              !  (18), t
770  ! Y(Ts)=Y=Muy+Sigy*Yb(Ts)!   (18)
780  ! PRESET PROBABILITIES: REMOVE 890-1040 AND INSERT 790-880
790  ! DATA .9,.99,.999,.9999,.99999,.999999,.9999999
800  ! READ P(*)                  !  (19)
810  ! FOR Ts=1 TO T              !  (19), t
820  ! P=P(Ts)
830  ! R=1-P^Rk                   !  (58)
840  ! IF M>2 THEN 870
850  ! Y(Ts)=Y=-LOG(R)            !  (58)
860  ! GOTO 880
870  ! Y(Ts)=Y=M*(R^(-1/M)-1)  !   (9) & (67)
880  ! Yb(Ts)=(Y-Muy)/Sigy        !  (12)
890    R=1-P1^Rk                  !  (9) & (67)
900    IF M>2 THEN 930
910    Y1=-LOG(R)                 !  (58)
920    GOTO 940
930    Y1=M*(R^(-1/M)-1)          !  (67)
940    Ay=10^(Dby/10)             !  (10)
950    Ytc=Y1*Ay                  !  (10)
960    Dely=(Ytc-Y1)/(T-1)        !  (11)
970    FOR Ts=1 TO T              !  (11), t
980    Y(Ts)=Y=Y1+Dely*(Ts-1)  !  (11)
990    Yb(Ts)=(Y-Muy)/Sigy        !  (12)
1000   IF M>2 THEN 1030
1010   R=EXP(-Y)                  !  (57)
1020   GOTO 1040
1030   R=(1+Y/M)^(-M)             !  (13) & (66)
1040   P(Ts)=(1-R)^K              !  (66)
1050   NEXT Ts
```

```
1060    Q1=1-P(1)^R1              !  (84)
1070    Qtc=1-P(T)^R1             !  (84)
1080    IF N>2 THEN 1120
1090    Z1=-LOG(Q1)               !  (60)
1100    Ztc=-LOG(Qtc)             !  (60)
1110    GOTO 1150
1120    Rn=-1/N                   !  (84)
1130    Z1=N*(Q1^Rn-1)            !  (21) & (84)
1140    Ztc=N*(Qtc^Rn-1)          !  (21) & (84)
1150    Delz=(Ztc-Z1)/(U-1)       !  (24)
1160    FOR Us=1 TO U             !  (24), u
1170    Zp(Us)=Z=Z1+Delz*(Us-1)!  (24)
1180    Zbp(Us)=(Z-Muz)/Sigz      !  (26)
1190    IF N>2 THEN 1220
1200    Pp(Us)=(1-EXP(-Z))^L      !  (25) & (59)
1210    GOTO 1230
1220    Pp(Us)=(1-(1+Z/N)^(-N))^L ! (25) & (83)
1230    NEXT Us
1240    Dbz=10*LGT(Zp(U)/Zp(1))!  (23)
1250    PRINT " Y THRESHOLDS          NORMALIZED          PROBABILITIES"
1260    FOR Ts=1 TO T
1270    PRINT Y(Ts),Yb(Ts),P(Ts)
1280    NEXT Ts
1290    PRINT
1300    PRINT " Z THRESHOLDS          NORMALIZED          PROBABILITIES"
1310    FOR Us=1 TO U
1320    PRINT Zp(Us),Zbp(Us),Pp(Us)
1330    NEXT Us
1340    PRINT
1350    PRINT "Dbz =";Dbz
1360    PRINT
1361      PAUSE
1370    PRINT " Pd1 =";Pd1;"          Pd2 =";Pd2;"          Pd3 =";Pd3
1380    PRINT
1390    PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:"
1400    IF M>2 THEN 1530
1410    FOR Ts=1 TO T             !  (94), t
1420    Y=Y(Ts)
1430    Q=(1-EXP(-Y))^(K-1)       !  (95)
1440    D1=-LOG(1-(1-Pd1)/Q)      !  (94)
1450    Rt1=Y/D1-1                !  (94)
1460    D2=-LOG(1-(1-Pd2)/Q)
1470    Rt2=Y/D2-1
1480    D3=-LOG(1-(1-Pd3)/Q)
1490    Rt3=Y/D3-1
1500    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1510    NEXT Ts
1520    GOTO 1660
```

```
1530    Rm=-1/M                        !  (99)
1540    FOR Ts=1 TO T                  !  (99), t
1550    Ym=Y(Ts)/M                     !  (99) & (100)
1560    Q=(1+Ym)^(-M)                  !  (100)
1570    Q=(1-Q)^(K-1)                  !  (100)
1580    D1=(1-(1-Pd1)/Q)^Rm-1          !  (99)
1590    Rt1=Ym/D1-1                    !  (99)
1600    D2=(1-(1-Pd2)/Q)^Rm-1
1610    Rt2=Ym/D2-1
1620    D3=(1-(1-Pd3)/Q)^Rm-1
1630    Rt3=Ym/D3-1
1640    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1650    NEXT Ts
1660    PRINT
1670    PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:"
1680    IF N>2 THEN 1810
1690    FOR Us=1 TO U                  !  SIMILAR
1700    Z=Zp(Us)                       !  TO
1710    Q=(1-EXP(-Z))^(L-1)            !  (94)
1720    D1=-LOG(1-(1-Pd1)/Q)           !  &
1730    R1=Z/D1-1                      !  (95)
1740    D2=-LOG(1-(1-Pd2)/Q)
1750    R2=Z/D2-1
1760    D3=-LOG(1-(1-Pd3)/Q)
1770    R3=Z/D3-1
1780    PRINT 10*LGT(R1),10*LGT(R2),10*LGT(R3)
1790    NEXT Us
1800    GOTO 1940
1810    Rn=-1/N                        !  SIMILAR
1820    FOR Us=1 TO U                  !  TO
1830    Zn=Zp(Us)/N                    !  (99)
1840    Q=(1+Zn)^(-N)                  !  &
1850    Q=(1-Q)^(L-1)                  !  (100)
1860    D1=(1-(1-Pd1)/Q)^Rn-1
1870    R1=Zn/D1-1
1880    D2=(1-(1-Pd2)/Q)^Rn-1
1890    R2=Zn/D2-1
1900    D3=(1-(1-Pd3)/Q)^Rn-1
1910    R3=Zn/D3-1
1920    PRINT 10*LGT(R1),10*LGT(R2),10*LGT(R3)
1930    NEXT Us
1940    PRINT
1950    END
```

P1 = .85	Dby = 10	T = 7	U = 15
M = 8	K = 12	N = 16	L = 24
Muy = 3.94631506068		Sigy = 2.09538979024	
Muz = 4.32438211761		Sigz = 1.69328544323	

Y THRESHOLDS	NORMALIZED	PROBABILITIES
5.7085601983	.841010653879	.85
14.2714004957	4.92752493268	.996679075641
22.8342407932	9.01403921147	.999753631023
31.3970810906	13.1005534903	.999965311792
39.9599213881	17.1870677691	.99999280763
48.5227616855	21.2735820479	.999998067526
57.085601983	25.3600963267	.999999374796

Z THRESHOLDS	NORMALIZED	PROBABILITIES
5.86721821227	.911149446675	.85
7.70938566512	1.99907437995	.956529647638
9.55155311797	3.08699931323	.98667504332
11.3937205708	4.1749242465	.99560662072
13.2358880237	5.26284917978	.998447087595
15.0780554765	6.35077411306	.999415549265
16.9202229294	7.43869904633	.999767363279
18.7623903822	8.52662397961	.999902645376
20.6045578351	9.61454891288	.999957385298
22.4467252879	10.7024738462	.999980574274
24.2888927408	11.7903987794	.999990813211
26.1310601936	12.8783237127	.999995507461
27.9732276465	13.966248646	.999997734746
29.8153950993	15.0541735793	.999998825244
31.6575625522	16.1420985125	.999999374796

Dbz = 7.32045232969

Pd1 = .5	Pd2 = .9	Pd3 = .99

SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:

7.18029480646	16.5239686654	26.8808804974
12.6997498763	21.2426978346	31.503743108
14.8466399857	23.3091924052	33.5584316626
16.269595645	24.6986419313	34.9428284925
17.3390310658	25.7492651765	35.9905902914
18.1963923488	26.5945353622	36.8340131363
18.9120679996	27.3017783991	37.5399641554

SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:

7.40601673703	16.6306689407	26.9721139815
9.64374630239	18.3703967446	28.6551266507
10.8741086655	19.4515609092	29.7168048237
11.7563696979	20.2655149832	30.5212118605
12.4659011251	20.9347216111	31.1845531717
13.0680388962	21.5092203112	31.7549390039
13.5941676551	22.0147264365	32.2573356581
14.0624498591	22.4668270369	32.7069765459
14.4847595558	22.8760203338	33.1141652155
14.8694755601	23.2498575248	33.4863332091
15.2228081965	23.5940083337	33.8290709181
15.5495188303	23.9128603228	34.1467104786
15.8533503185	24.2098871812	34.4426853709
16.1373031597	24.4878[illegible]96233	34.7197662984
16.4038217386	24.749160188	34.9802228898

REFERENCES

[1] I. S. Gradshteyn and I. M. Ryzhik, **Table of Integrals, Series, and Products**, Academic Press, Inc., New York, NY, 1980.

[2] **Handbook of Mathematical Functions**, U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series, Number 55, U. S. Government Printing Office, Washington, DC, June 1964.

[3] A. H. Nuttall and A. F. Magaraci, **Signal-to-Noise Ratios Required for Short-Term Narrowband Detection of Gaussian Processes**, NUSC Technical Report 4417, Naval Underwater Systems Center, New London, CT, 20 October 1972.

[4] M. G. Kendall and A. Stuart, **The Advanced Theory of Statistics, Volume 1, Distribution Theory**, Third Edition, Hafner Publishing Company, New York, NY, 1969.

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
Admiralty Underwater Weapons Establishment, England	
Library	1
Center for Naval Analyses	1
Coast Guard Academy	
Prof. J. J. Wolcin	1
David Taylor Research Center, Bethesda, MD	
Commander	1
H. Chaplin	1
L. Becker	1
W. Phillips (Code 1932)	1
David Taylor Research Center, Annapolis	
P. Prendergast (Code 2744)	1
Defense Nuclear Agency, RAAE	
J. Meyers	1
Defence Research Center, Australia	
Library	1
Defence Research Establishment Atlantic, Nova Scotia	
B. E. Mackey (Library)	1
Defence Research Establishment Pacific, British Columbia	
Dr. D. J. Thomson	1
Defence Science Establishment, HMNZ Dockyard, New Zealand	
Director	1
Defence Scientific Establishment, New Zealand	
Dr. L. H. Hall	1
Defense Advanced Research Projects Agency	
Commanding Officer	1
A. W. Ellinthorpe	1
Defense Intelligence Agency	1
Defense Technical Information Center	12
Dept. of Science & Industrial Research, New Zealand	
M. A. Poletti	1
National Radio Astronomy Observatory	
F. Schwab	1
National Security Agency	
Dr. J. R. Maar (R51)	1
Naval Air Development Center	
Commander	1
A. Witt	1
T. Madera	1
L. Allen (Code 50)	1
Naval Air Systems Command	
NAIR-93	1
Naval Coastal Systems Center	
Commanding Officer	1
D. Skinner	1
E. Linsenmeyer	1
Naval Environmental Prediction Research Facility	1
Naval Intelligence Command	1
Naval Oceanographic Office	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Naval Oceanographic & Atmospheric Research Laboratory, CA	
M. J. Pastore	1
Naval Oceanographic & Atmospheric Research Laboratory, MS	
Commanding Officer	1
R. Wagstaff (Code 245)	1
B. Adams	1
E. Franchi	1
R. Fiddler (Code 245)	1
Naval Ocean Systems Center, Hawaii	1
Naval Ocean Systems Center, San Diego	
Commanding Officer	1
J. M. Alsup (Code 635)	1
J. Silva	1
D. Hanna	1
P. Nachtigall	1
W. Marsh	1
C. Persons	1
Naval Personnel Research & Development Center	1
Naval Postgraduate School	
Superintendent	2
Prof. C. W. Therrien (Code 62 TI)	1
Naval Research Laboratory, Orlando, USRD	1
Naval Research Laboratory, Washington	
Commanding Officer	1
Dr. P. B. Abraham (Code 5131)	1
W. F. Gabriel (Code 5370)	1
A. A. Gerlach	1
N. Yen (Code 5130)	1
D. Bradley	1
S. Sachs	1
D. Steiger	1
E. Wald (Code 5150)	1
Naval Sea Systems Command	
SEA-00; -63; -63D; -63X; -92R; PMS-402	6
Naval Surface Warfare Center	
J. Gray (Code G71)	1
Naval Surface Weapons Center, Dahlgren, VA	
Commander	1
H. Crisp	1
D. Phillips	1
T. Ryczek	1
Naval Surface Weapons Center, White Oak Lab.	1
M. Strippling	1
Naval Surface Weapons Center, Fort Lauderdale	1
Naval Technical Intelligence Center	
Commanding Officer	2
D. Rothenberger	1
Naval Underwater Systems Center, West Palm Beach	
Officer-in-Charge	1
Dr. R. M. Kennedy (Code 3802)	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Naval Weapons Center	1
Norwegian Defence Research Establishment	
Dr. J. Glattetre	1
Office of the Chief of Naval Research, Arlington, VA	
OCNR-00; -10; -11; -12; -13; -20; -21; -22; -23 (3)	11
N. L. Gerr (Code 1111)	1
A. Wood	1
D. Johnson	1
SACLANT Undersea Research Center	
Dr. J. P. Ianniello	1
Dr. S. Stergiopolous	1
Prof. G. Tacconi	1
Library	1
Sonar and Surveillance Group, RANRL, Australia	1
Space & Naval Warfare System Command	
SPAWAR-00; -04; -005; PD-80; PMW-181	5
L. Parrish	1
R. Cockerill	1
U.S. Air Force, Alabama	
Air University Library	1
U.S. Coast Guard Research & Development Center	
Library	1
U.S. Department of Commerce, NTIA/ITS	
Dr. A. D. Spaulding	1
Weapons Systems Research Laboratory, Australia	
HASC	1
HSPC	1
Brown University	
Documents Library	1
Canberra College of Advanced Education	
P. Morgan	1
Concordia University, Quebec	
Prof. J. Krolik	1
Dalhousie University	
Dr. B. Ruddick	1
Drexel University	
Prof. S. B. Kesler	1
Harvard University	
Gordon McKay Library	1
Indian Institute of Technology, India	
Dr. K. M. M. Prabhu	1
Johns Hopkins University, Applied Physics Laboratory	
Director	1
J. C. Stapleton	1
Kansas State University	
Prof. B. Harms	1
Lawrence Livermore National Laboratory	
Director	1
L. C. Ng	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Los Alamos National Laboratory	1
Marine Biological Laboratory, Woods Hole	1
Marine Physical Laboratory, Scripps	1
Massachusetts Institute of Technology	
Prof. A. B. Baggaroer	1
Barker Engineering Library	1
Northeastern University	
Prof. C. L. Nikias	1
Penn State University, Applied Research Laboratory	
Director	1
F. W. Symons	1
R. Hettche	1
E. Liszka	1
Purdue University	
Prof. N. Srinivasa	1
Royal Military College of Canada	
Prof. Y. T. Chan	1
Rutgers University	
Prof. S. Orfanidis	1
San Diego State University	
Prof. F. J. Harris	1
Sandia National Laboratory	
Director	1
J. Claasen (315)	1
Simon Fraser University	
Dr. E. F. Velez	1
Southeastern Massachusetts University	
Prof. C. H. Chen	1
State University of New York	
Prof. M. Barkat	1
Tel- Aviv University, Israel	
Prof. E. Weinstein	1
United Engineering Center	
Engineering Societies Library	1
University of Alberta, Canada	
K. L. Yeung	1
University of Auckland	
Dr. M. D. Johns	1
University of California, San Diego	
Prof. C. W. Helstrom	1
University of Colorado	
Prof. L. L. Scharf	1
University of Connecticut	
Prof. C. H. Knapp	1
Wilbur Cross Library	1
University of Florida	
Prof. D. C. Childers	1
University of Illinois	
Dr. D. L. Jones	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
University of Michigan	
Communications & Signal Processing Laboratory	1
W. J. Williams	1
University of Minnesota	
Prof. M. Kaveh	1
University of Queensland	
Dr. B. Boashash	1
University of Rhode Island	
Prof. G. F. Boudreaux-Bartels	1
Prof. S. M. Kay	1
Prof. D. Tufts	1
Library	1
University of Rochester	
Prof. E. Titlebaum	1
University of Southern California	
Prof. W. C. Lindsey	1
Dr. A. Polydoros	1
University of Strathclyde, Scotland	
Prof. T. S. Durrani	1
University of Technology, England	
Prof. J. W. R. Griffiths	1
University of Texas, Applied Research Laboratory	1
University of Washington	
Applied Physics Laboratory	1
Prof. D. W. Lytle	1
Dr. R. C. Spindel	1
Villanova University	
Prof. M. G. Amin	1
Woods Hole Oceanographic Institution	
Director	1
Dr. E. Weinstein	1
Yale University	
Prof. A. Nehorai	1
Prof. P. M. Schultheiss	1
Prof. F. Tuteur	1
Kline Science Library	1
Applied Seismic Group	
R. Lacoss	1
A&T, North Stonington, CT	
H. Jarvis	1
A&T, Arlington, VA	
D. L. Clark	1
BB&N, Arlington, VA	
Dr. H. Cox	1
BB&N, Cambridge, MA	
H. Gish	1
BB&N, New London, CT	
Dr. P. G. Cable	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Bell Communications Research	
J. F. Kaiser	1
D. Sunday	1
Berkeley Research Associates	
Steven W. McDonald	1
Cogent Systems, Inc.	
J. P. Costas	1
Diagnostic/Retrieval Systems, Inc.	
J. Williams	1
EDO Corporation	
M. Blanchard	1
EG&G	
D. Frohman	1
General Electric Company, Moorestown, NJ	
Dr. M. R. Allen	1
H. Urkowitz	1
General Electric Company, Philadelphia, PA	
T. J. McFall	1
General Electric Company, Pittsfield, MA	
R. W. Race	1
General Electric Company, Syracuse, NY	
Dr. A. M. Vural	1
D. Winfield	1
Harris Scientific Services	
B. Harris	1
Hughes Aircraft Company, Buena Park, CA	
T. E. Posch	1
IBM	
G. L. Demuth	1
Interstate Electronics Corporation	
R. O. Nielsen (Dept. 8011)	1
Kildare Corporation	
Dr. R. Mellen	1
Lincom Corporation	
Dr. T. A. Schonhoff	1
Magnavox Elec. Systems Company	
R. Kenefic (Dept. 525)	1
MSB Systems, Inc.	
A. Winder	1
Nichols Research Corporation	
Dr. T. L. Marzetta	1
Orincon Corporation	
S. L. Marple	1
Dr. J. W. Young	1
Prometheus, Inc., Newport, RI	
M. J. Barrett	1
Prometheus, Inc., Sharon, MA	
Dr. J. S. Byrnes	1
RAN Research Lab, Australia	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Raytheon, Portsmouth, RI	
J. Bartram	1
R. Conner	1
S. S. Reese	1
Rockwell International	
L. T. Einstein	1
Dr. D. F. Elliott	1
SAIC, Falls Church, VA	
Dr. P. Mikhalevsky	1
SAIC, New London	
Dr. F. DiNapoli	1
SIMRAD SUBSEA A/S, Naval Systems Div.	
E. B. Lunde	1
Sperry Corporation, Defense Marine Systems Unit	1
Toyon Research Corporation	
M. L. Van Blaricum	1
Tracor	
Dr. T. J. Leih	1
TRW, Fairfax, VA	
R. Prager	1
G. Maher	1
TRW, New London, CT	
D. Sturim	1
Welse Sistemi Subacquei, Genova, Italy	
Dr. H. Van Asselt	1
Westinghouse Electric Corporation, Annapolis, MD	
H. Newman	1
Dr. H. L. Price	1
Westinghouse Electric Corporation, Baltimore, MD	
R. Park	1
Westinghouse Electric Corporation, Waltham, MA	
R. B. Kennedy	1
Assard, G. L.	1
Beidas, B.	1
Bendat, Dr. J. S.	1
Bleistein, Dr. N.	1
Cohen, Dr. L.	1
Hahn, W. R.	1
Lloyd, L.	1
Maltz, F.	1
Middleton, Dr. D.	1
Nash, H. E.	1
Nicholson, D.	1
O'Brien, W.	1
Papoutsanis, P.D.	1
Pohler, R. F.	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Price, Dr. R.	1
Raisbeck, Dr. G.	1
Richter, W.	1
Schulkin, Dr. M.	1
Urick, R. J.	1
Von Winkle, Dr. W. A.	1
Werbner, A.	1
Wilson, Dr. J. H.	1